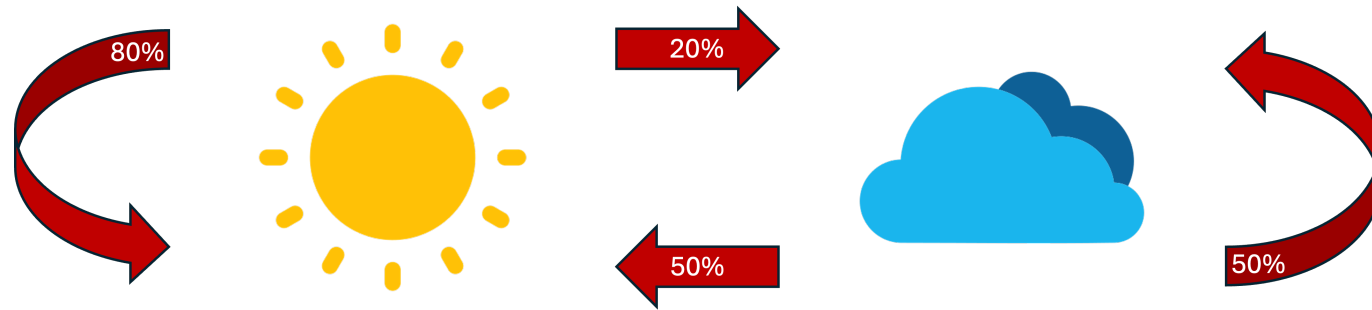


Markov Chain Convergence and Rubik's Cube Scrambling

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Joint work with Tomas Rokicki and Hillary Yang

What is the weather like?

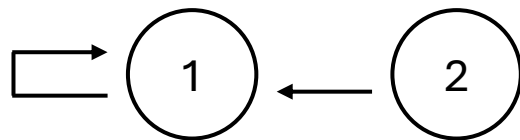


How fast does $P(X_n = \text{☀})$ converge to $P(X_\infty = \text{☀})$?

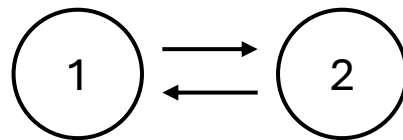
$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix}, \quad \pi = (5/7 \quad 2/7), \quad \nu P^n \rightarrow \pi, \quad n \rightarrow \infty$$

When does a finite-state MC converge?

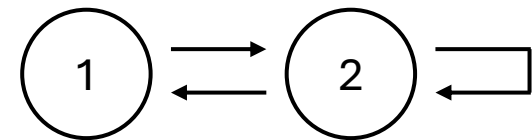
- Irreducibility: $\forall x, y, \exists m, \text{ s.t. } p^m(x, y) > 0$.
- Aperiodicity: $\forall x, \text{ gcd}\{m : p^m(x, x) > 0\} = 1$.
- An irreducible aperiodic finite-state Markov chain is positive recurrent and converges to a unique stationary distribution exponentially fast.



reducible



irreducible



aperiodic

periodic

How is convergence measured?

- Total variation distance:

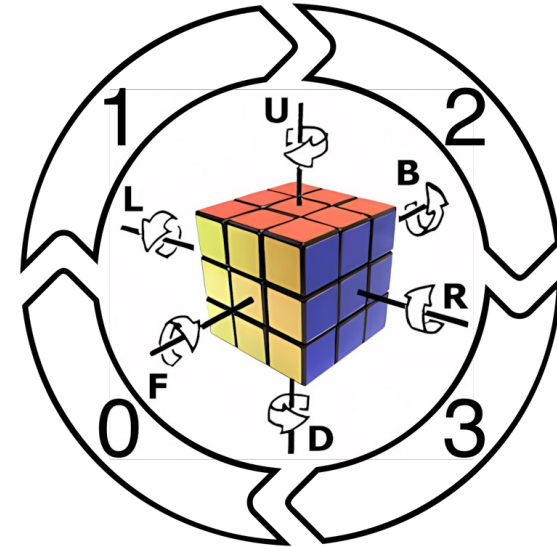
$$\begin{aligned} \text{TV}(X_n, X_\infty) &= \sup_{0 \leq h \leq 1} |\mathbb{E}h(X_n) - \mathbb{E}h(X_\infty)| \\ &= (1/2) \sum_{x \in \mathcal{X}} |P(X_n = x) - P(X_\infty = x)| \end{aligned}$$

- Mixing time:

$$t_{\text{mix}}(\epsilon) = \min\{n : \text{TV}(X_n, X_\infty) < \epsilon\}, \quad t_{\text{mix}} = t_{\text{mix}}(1/4)$$

Rubik's Cube

- U:up, D:down, F:front, B:back, L:left, R:right;
- 1:90°, 2:180°, 3:270° (clockwise);
- U1, U2, U3, D1, D2, D3, F1, F2, F3, B1, B2, B3, L1, L2, L3, R1, R2, R3;
- $|\mathcal{X}| = 43,252,003,274,489,856,000$;
- Let f be a move sampled uniformly at random from the 18 options above. The scrambling process is $X_{n+1} = f_{n+1}(X_n)$ where f_{n+1} 's are iid copies of f .
- Exercise: i) it is irreducible, ii) it is aperiodic, iii) its equilibrium is uniform.



Main result

Rubik's Cube scrambling requires at least 26 random moves, i.e., $t_{\text{mix}} \geq 26$.

Proof

distribution over
 4.3×10^{19} states



distribution over
 $\{0, 1, 2, \dots, 20\}$



optimally solving
2M random cubes



$$\text{TV}(X_{25}, X_{\infty}) \geq \text{TV}(d_o(X_{25}), d_o(X_{\infty})) = 0.286329 \pm 0.000697 > 0.25$$



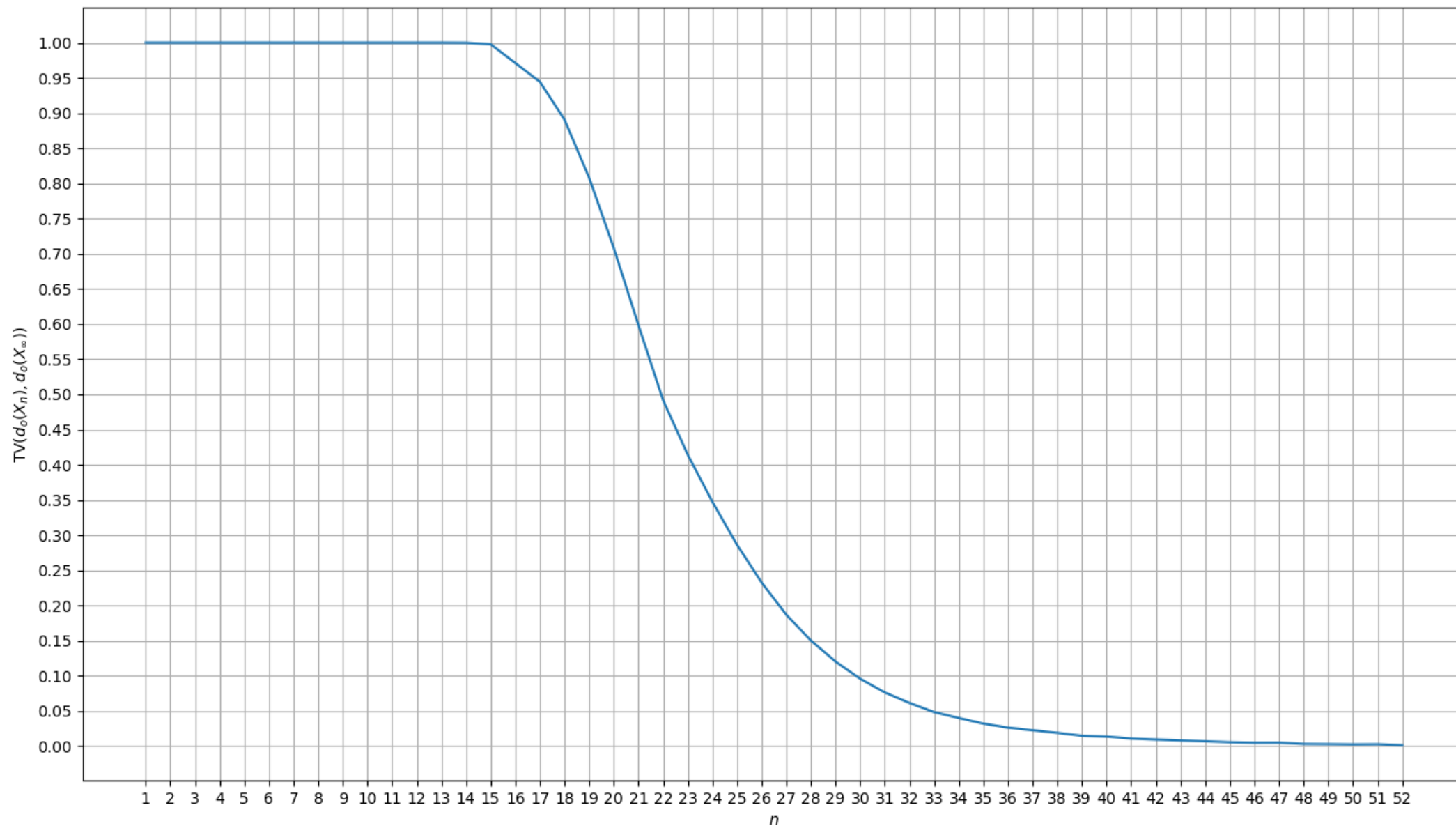
exercise



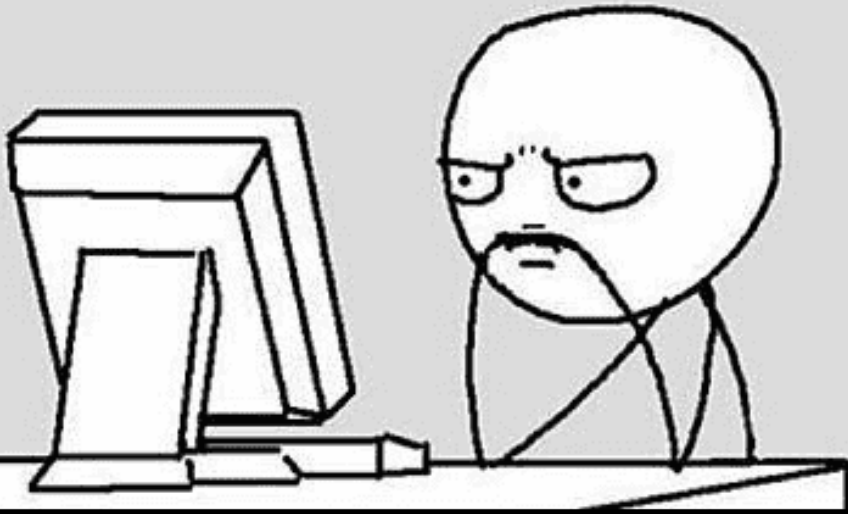
distance to
the solved state



bootstrap

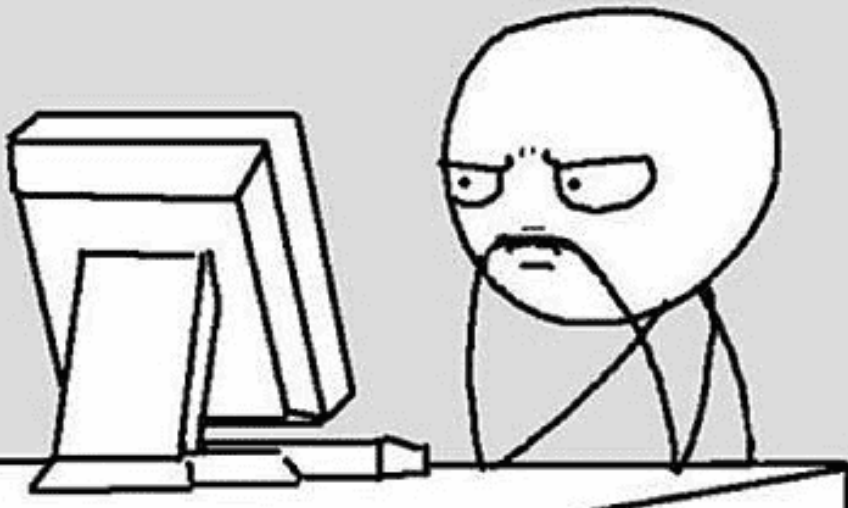


It doesn't work..... why?

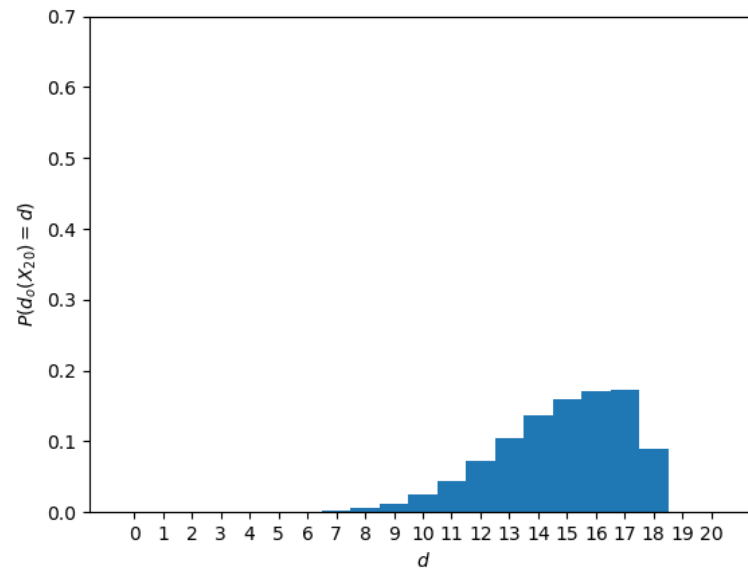
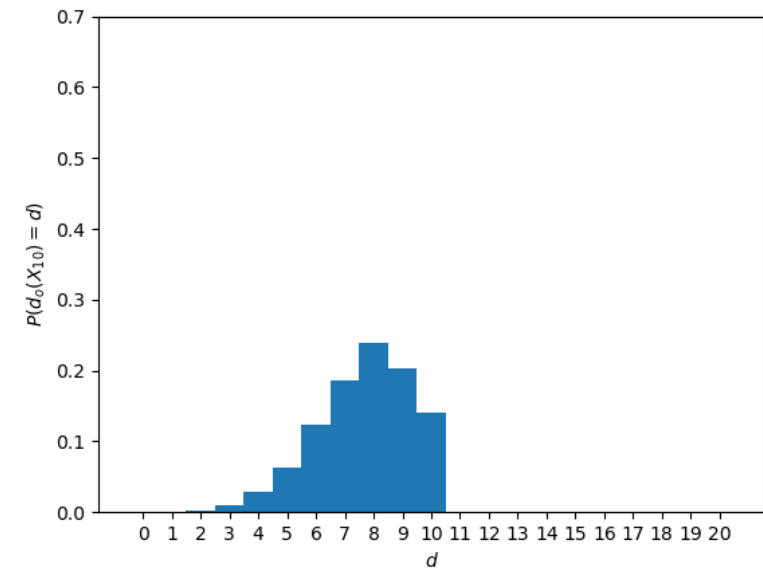


- embedded 2x2x2 pocket cube
- 3,674,160 states
- mixing time 19

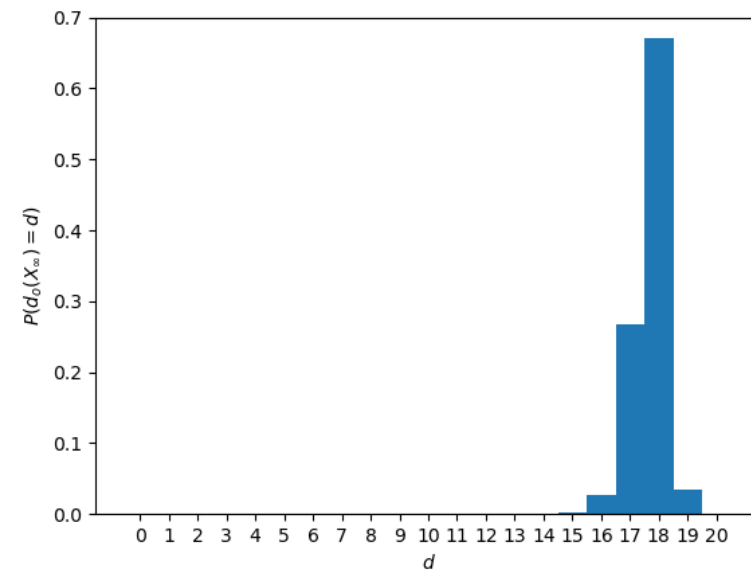
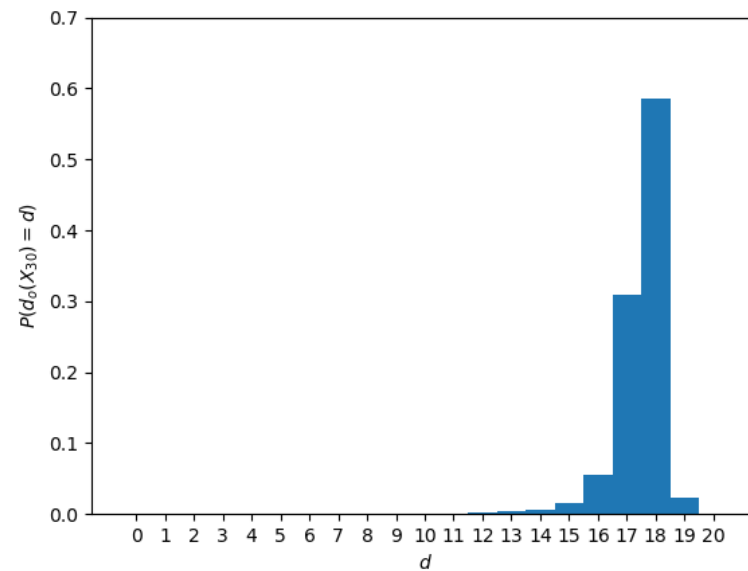
It works..... why?



- distance to the solved state
- 21 states
- mixing time 26



moving instead of mixing



Thank You



<https://arxiv.org/abs/2410.20630>

Rubik's Cube Scrambling Requires at Least 26 Random Moves